Close operator algebras

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A metric on subalgebras of $\mathcal{B}(\mathcal{H})$

Kadison-Kastler 1972

Definition

Let A, B be C^*-subalgebras of $\mathcal{B}(\mathcal{H})$. The Kadison-Kastler distance $d(A, B)$ is the infimum of $\gamma > 0$ such that for all operators x in the unit ball of one algebra, there exists y in the unit ball of the other algebra with $\|x - y\| < \gamma$.

Theme of the talk

What can be said when $d(A, B)$ is small?

- Aim: Give survey of what is known.
- See similarities and differences between C^*-algebra and von Neumann algebra settings.
- Establish connections to similarity and derivation problems.

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EASY CONSTRUCTION

For a unitary u, $d(A, uAu^*) \leqslant 2\|u - 1_{\mathcal{H}}\|$.

Is this the only way of constructing a close pair of operator algebras?

MORE GENERALLY, WE HAVE A RANGE OF QUESTIONS

Suppose $A, B \subset \mathcal{B}(\mathcal{H})$ have $d(A, B)$ small.

1. Must A and B share the same properties and invariants?
2. Must A and B be *-isomorphic?
3. Must A and B be spatially isomorphic? Can one find a unitary implementing a spatial isomorphism in $(A \cup B)''$?
4. Is there a unitary $u \approx 1_{\mathcal{H}}$ with $uAu^* = B$?

- Kadison-Kastler conjectured **??**. Open for separable C^*-algebras
- **??** is open for von Neumann algebras. Fails for separable C^*-algebras.

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QUESTIONS ABOUT CLOSE OPERATOR ALGEBRAS

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THEOREM (KADISON-KASTLER 1972)

Close von Neumann algebras have the same type decompositions.

Precisely, suppose:

- M, N are von Neumann algebras on $\mathcal{H}$ with $d(M, N)$ sufficiently small.
- $p_{\mathrm{I}_n}, p_{\mathrm{II}_1}, p_{\mathrm{II}_\infty}, p_{\mathrm{III}}$ be the central projections in M onto the parts of types I_n, II_1, II_∞ and III respectively.
- $q_{\mathrm{I}_n}, q_{\mathrm{II}_1}, q_{\mathrm{II}_\infty}, q_{\mathrm{III}}$ corresponding projections for N.

Then each $\|p_j - q_j\|$ is small.

They also show that if $d(M, N)$ is small ($< 1/10$), then M is a factor if and only if N is a factor.

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SOME C^*-INVARIANTS

THEOREM (PHILLIPS 1974)

Suppose A and B are sufficiently close C^*-algebras. Then

- *A and B have isomorphic and close ideal lattices.*
- *This takes primitive ideals to primitive ideals and is a homeomorphism for the hull-kernel topology.*
- *A is type I if and only if B is type I.*

By isomorphic and close ideal lattices, we mean that there is a lattice isomorphism $A \trianglerighteq I \mapsto \theta(I) \trianglelefteq B$ such that $d(I, \theta(I))$ is small for all I.

COROLLARY

If $d(A, B)$ is sufficiently small and A is abelian, then $A \cong B$.

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Near Containments

Christensen 1980

Definition

For $A, B \subset \mathcal{B}(\mathcal{H})$ write $A \subseteq_\gamma B$ if given $x \in A$, there exists $y \in B$ such that $\|x - y\| \leqslant \gamma\|x\|$. In this case say A is γ-contained in B.

Similar range of questions:

1. Must a sufficiently small near containment $A \subset B$ give rise to an embedding $A \hookrightarrow B$?
2. If so, can an embedding $\theta : A \hookrightarrow B$ with $\|\theta - \iota\|$ small be found?
3. Must a sufficiently small near containment arise from a small unitary conjugate of a genuine inclusion?

A CB-VERSION OF THE METRIC

- It's natural to take matrix amplifications of operator algebras
- $A \subset \mathcal{B}(\mathcal{H})$, gives $M_n(A) \subseteq M_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^n)$.

DEFINITION

Given $A, B \subset \mathcal{B}(\mathcal{H})$, define

$$d_{cb}(A,B) = \sup_n (M_n(A), M_n(B)).$$

Similarly $A \subseteq_{cb,\gamma} B$ iff $M_n(A) \subseteq_\gamma M_n(B)$ for all n.

THEOREM (KHOSHKAM 1984)

Suppose A, B are C^*-algebras with $d_{cb}(A,B) < 1/3$. Then $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.

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ARVESON'S DISTANCE FORMULA

Let $A \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra and $T \in \mathcal{B}(\mathcal{H})$. Then

$$d(T, A') = \|\mathrm{ad}(T)|_A\|_{cb}/2.$$

Here $\mathrm{ad}(T)|_A$ is the spatial derivation $x \mapsto [T, x] = Tx - xT$.

CONSEQUENCE

- $A \subseteq_{\gamma,cb} B \implies B' \subseteq_{\gamma,cb} A'$

TWO QUESTIONS

1. Are d and d_{cb} locally equivalent? i.e. for each A is there some K_A such that $d_{cb}(A,\cdot) \leqslant K_A d(A,\cdot)$?
2. How does commutation behave in the metric d?

The CB-metric and commutants

Arveson's distance formula

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QUESTION (KADISON '54)

Let A be a C^*-algebra. Is every bounded homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$ similar to a $*$-homomorphism?

- Still open. If yes, say A has the similarity property.
- Yes if A has no bounded traces, A is nuclear.
- For II_1 factors M, yes when M has Murray and von Neumann's property Γ.

REFORMULATION USING CHRISTENSEN, KIRCHBERG

Let A be a C^*-algebra. Then A has the similarity property if and only if there exists a constant $K > 0$ such that for every representation $\pi : A \to \mathcal{B}(\mathcal{H})$, we have $\|\mathrm{ad}(T)|_{\pi(A)}\|_{cb} \leq K \|\mathrm{ad}(T)|_{\pi(A)}\|, \quad T \in \mathcal{B}(\mathcal{H})$.

- If A has SP, then $\exists K$ such that $A \subseteq_\gamma B \implies B' \subseteq_{cb, K\gamma} A'$.

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When are d_{cb} and d equivalent?

Theorem (Christensen, Sinclair, Smith, W))

Suppose A is a C^*-algebra with the similarity property. Then there exists $\gamma_0 > 0$ such that if $d(A, B) < \gamma_0$, then B has the similarity property.

- γ_0 depends only on how well A has the similarity property;
- Also obtain quantitative estimates on how well B has similarity property.

Corollary

If A has similarity property, then $\exists C > 0$ such that $d_{cb}(A, B) \leqslant Cd(A, B)$ for all B and so if $d(A, B)$ small, then $K_*(A) \cong K_*(B)$.

In fact this characterises the similarity property for A. Further, the similarity problem has a positive answer if and only the map $A \mapsto A'$ is continuous on C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ (for a separable infinite dimensional Hilbert space).

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More invariants and properties

Proposition (Christensen, Sinclair, Smith, W.)

Suppose $d(A, B) < 1/14$. Then A has real rank zero iff B has real rank zero.

The definition of real rank zero (the invertible self-adjoints are dense in the self-adjoints) wasn't very helpful. Used every hereditary subalgebra has an approximate unit of projections reformulation.

Question

What about higher values of the real rank, stable rank? It's unknown whether stable rank one transfers to sufficiently close algebras.

Theorem (Christensen, Raeburn-Taylor)

Let M and N be sufficiently close von Neumann algebras. Then M is injective if and only if N is injective. Similarly for nuclear C^*-algebras.

What about exactness?

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What about exactness?

THEOREM (PERERA, TOMS, W, WINTER)

Suppose $d_{cb}(A, B) < 1/42$. Then A and B have isomorphic Cuntz semigroups.

THEOREM

If $d_{cb}(A, B)$ is sufficiently small, and A is unital, then A and B have the same Elliott invariant.

This uses Khoskham's work, to get an isomorphism between K-theories, a method for transferring trace spaces from CSSW, then the Cuntz semigroup result (which gives a method for transferring quasi-trace spaces in a homeomorphic fashion, extending the map at the level of traces from CSSW).

MORE QUESTIONS

What about Ext, KK-theory, the UCT?

More on invariants

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Tensorial Absorption

- Tensorial absorption a key theme in operator algebras, since Connes showed that an injective II_1 factor M is McDuff, i.e. $M \cong M \overline{\otimes} R$, where R is the hyperfinite II_1 factor.

Theorem (Cameron, Christensen, Sinclair, Smith, W, Wiggins)

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Theorem (Christensen, Johnson, Raeburn-Taylor, 1977)

Suppose M and N are von Neumann algebras, with $d(M,N)$ sufficiently small and M injective. Then there exists a unitary $u \in (M \cup N)''$ such that $uMu^* = N$ and $\|u-1\| \leqslant O(d(M,N)^{1/2})$.

This gives the strongest form of the conjecture for injective von Neumann algebras. Also:

Theorem (Christensen 1980)

Suppose $M \subseteq_\gamma N$ for γ sufficiently small, where M is an injective von Neumann algebra. Then there exists a unitary $u \in (M \cup N)''$ with $uMu^* \subseteq N$ and $\|u-1\| \leqslant 150\gamma$.

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- Consider a ucp map $\Phi : \mathcal{B}(\mathcal{H}) \to N$ with $\Phi|_N = \mathrm{id}_N$.
- This is almost multiplicative on M, i.e. $\Phi(xy) \approx \Phi(x)\Phi(y)$.
- Find (using injectivity) a *-homomorphism $\tilde{\Phi} : M \to N$ close to Φ. One way to do this, is to do it for finite dimensional subalgebras of M with constants independent of the choice of subalgebra, then take a weak*-limit point.

Subsequently, Johnson extensively studied these ideas. He called a pair of Banach algebras (A, B) AMNM, if every almost multiplicative map $T : A \to B$ is near to a multiplicative map $S : A \to B$.

- (A, B) AMNM, whenever A an amenable Banach algebra, and B a dual Banach algebra.
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For $\epsilon > 0$, there exist two faithful representations of $C([0,1],\mathcal{K})$ on H with images A, B s.t. $d(A, B) < \epsilon$, yet any isomorphism $\theta : A \to B$ has $\|\theta(x) - x\| \geqslant \|x\|/70$ for some $x \in A$.

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AMNM problems in point norm topology

Idea

The uniform topology isn't the right topology for maps between C^*-algebras. Use the point norm-topology instead.

Point norm AMNM:

Given C^*-algebra A, a finite subset X of the unit ball of A and $\varepsilon > 0$, does there exist a Y such that cpc maps $T_Y : A \to B$ which are almost multiplicative on Y are close to a linear map $T_{X,\varepsilon} : A \to B$ with

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Let $\mathcal{H}$ be a separable Hilbert space and let $A, B \subset \mathcal{B}(\mathcal{H})$ be C^*-algebras. Suppose A is separable and nuclear and $d(A, B)$ is sufficiently small. Then $A \cong B$.

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Which C^*-algebras A have the property that when $d(A,B)$ is sufficiently small, there exists an isomorphism $\theta : A \to B$ with $\sup_{x\in A,\|x\|\leqslant 1} \|\theta(x) - x\|$ small?

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Recall that if $M \subseteq_\gamma N$ and M is injective, then there exists a unitary $u \approx 1$ with $uMu^* \subseteq N$.

Theorem (Hirshberg, Kirchberg, W '11)

Let A be separable and nuclear and suppose $A \subseteq_\gamma B$ for $\gamma < 10^{-6}$. Then $A \hookrightarrow B$.

Key ingredients

- A strengthening of the completely positive approximation property (due to Kirchberg) for nuclear C^*-algebras: the approximating maps can be taken to be convex combinations of cpc order zero maps.
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Consider a free, ergodic, probability measure preserving action $\alpha : \Gamma \curvearrowright (X, \mu)$ of a discrete group Γ and form the crossed product

$$L^\infty(X) \rtimes_\alpha \Gamma.$$

This is a II_1 factor, generated by $A = L^\infty(X)$ and unitaries $(u_g)_{g\in\Gamma}$ satisfying

$$u_g f u_g^* = f \circ \alpha_g^{-1}, \quad u_g u_h = u_{gh}, \quad g, h \in \Gamma,\ f \in L^\infty(X).$$

Note:

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- First factorise $N = N_0 \overline{\otimes} R$, conjugating by a unitary so that the copies of R are identical.
- As M is McDuff, it has the similarity property. This enables us to transfer to and from a standard form.
- Use the embedding theorems for injective von Neumann algebras to embedd each $(L^\infty(X) \cup \{u_g\})''$ into N_0.
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Theorem (Cameron, Christensen, Sinclair, Smith, W, Wiggins '11)

Let $M_0 = L^\infty(X) \rtimes_\alpha \Gamma$ be as above and suppose Γ is a lattice in a semisimple Lie group of rank at least 2 with no hermitian factors (e.g. $SL_n(\mathbb{Z})$ for $n \geqslant 3$). Let $M = M_0 \overline{\otimes} R$. If $M, N \subset \mathcal{B}(\mathcal{H})$ has $d(M,N)$ is sufficiently small, then there exists a unitary $u \approx 1$ with $uMu^* = N$.

- First factorise $N = N_0 \overline{\otimes} R$, conjugating by a unitary so that the copies of R are identical.
- As M is McDuff, it has the similarity property. This enables us to transfer to and from a standard form.
- Use the embedding theorems for injective von Neumann algebras to embedd each $(L^\infty(X) \cup \{u_g\})''$ into N_0.
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